The Research Base for

Pre-Algebra
Algebra 1
Geometry
Algebra 2

Lois Edwards, Mathematics
Education Consultant

Macmillan/McGraw-Hill
Glencoe

The McGraw·Hill Companies

Table of Contents

The Research Base for *Pre-Algebra, Algebra 1, Geometry,* and *Algebra 2*

Lois Edwards, Mathematics Education Consultant

Introduction

Extensive efforts during development ensure that results of the latest research in mathematics education form the basis for Glencoe's *Pre-Algebra, Algebra 1, Geometry*, and *Algebra 2*.

Educational research provides information about what students are likely to learn in specific content areas at specific levels and under specific pedagogical conditions. It provides guidance for effective curriculum and instruction. Glencoe's authors have incorporated the results of educational research in mathematics into this curriculum, enabling teachers to provide their students with excellent learning experiences.

This white paper describes research results in several key areas of mathematics education: curriculum, educational principles, instructional strategies, and mathematical concepts and skills. It points out the Glencoe features that exemplify these research results and provides page numbers of selected examples. A comprehensive list of research references is included.

Curriculum

NCTM *Principles and Standards for School Mathematics*

The past two decades have seen an increased recognition of the importance of mathematics for every student and accompanying need for creating uniform national standards in mathematics education. The National Council of Teachers of Mathematics (NCTM) has led this movement.

Beginning in 1989, NCTM released publications on curriculum and evaluation, assessment, and professional standards, which have given focus, organization, and fresh ideas to efforts to improve mathematics education. In 2000, NCTM published the *Principles and Standards for School Mathematics*, a vision to guide educators toward continual improvement of mathematics education in classrooms, schools, and educational systems. It set forth important overall characteristics of mathematics programs (principles) and described the mathematical content that students should learn (standards). The *Principles and Standards for School Mathematics* are consistent with the best evidence on teaching and learning mathematics. They were written through a complex process that involved professional judgment, societal expectations, and research results.

Glencoe's mathematics series incorporates the key characteristics recommended by *Principles and Standards for School Mathematics* for effective curricula:

- Interconnected topical strands, such as algebra and geometry;
- Mathematical concepts that are organized and integrated so that students see how concepts are related;
- Foundational concepts such as equivalence, proportionality, function, and rate of change;
- Activities that develop mathematical thinking and reasoning skills, including making conjectures and developing deductive arguments;
- Experiences that demonstrate mathematics' usefulness in modeling and predicting real-world phenomena; and
- Guidance for teachers on the depth of study warranted at specific levels and when closure is expected.

Glencoe's *Pre-Algebra, Algebra 1, Geometry*, and *Algebra 2* series meets all six of the principles set forth in the *Principles and Standards for School Mathematics*.

- **Equity** Glencoe's texts encourage high achievement at all levels. Numerous teacher support materials provide activities for differentiated instruction, promotion of reading and writing skills, pacing for individual levels of achievement, and daily intervention opportunities.
- **Curriculum** Glencoe's curriculum is based on a detailed scope and sequence that ensures a continuum of mathematical learning that builds on prior knowledge.
- **Teaching** The comprehensive Teacher Editions provide mathematical background, teaching tips, resource management guides, and tips for new teachers.
- **Learning** The Teacher Editions include ways to build on students' prior knowledge and suggestions for dealing with common misconceptions. Students engage in both practice and exploratory activities.
- **Assessment** A full array of diagnostic, formative, and summative assessment is included.
- **Technology** The Student Editions offer opportunities to use graphing calculators, spreadsheets, and geometry software. Glencoe's Web site is constantly updated to meet the needs of students and teachers.

Glencoe's series also meets all of the *Principles and Standards for School Mathematics*' Content Standards. (For details, see the sections in this document on Instructional Strategies and Mathematical Concepts and Skills.)

Curriculum Characteristics

Research shows that a curriculum that enables students to reach high mathematics standards has the following characteristics:

- **Balance**—focuses on both conceptual understanding and procedural fluency;
- **Comprehensiveness**—includes all the important content strands of mathematics, as well as computation and other procedural skills;
- **Alignment with standards**—includes content and strategies which align with state and national standards, external assessments and instruction; and
- **Coordination and coherence within and across grades**—presents well-developed concepts that build on and connect with other concepts, both within and across grades (Apthorp, Bodrova, Dean, & Florian, 2001).

Research shows that student learning is affected by the written curriculum but also by teacher beliefs, teacher knowledge, teacher interpretation of curriculum, and classroom structures and norms (Stein et al., 2007).

Providing opportunities both for discovery and for practice improves student achievement. Students can learn both concepts and skills by solving problems (Grouws & Cebulla, 2000).

Focusing instruction on the meaningful development of mathematical ideas increases the level of student learning. It is important to emphasize meaning and how the idea, concept, or skill is connected to other mathematical ideas (Grouws & Cebulla, 2000).

Glencoe's *Pre-Algebra, Algebra 1, Geometry*, and *Algebra 2* exemplify these characteristics. Balanced instruction throughout the series emphasizes both understanding and fluency. Features include Key Concept boxes with multiple representations, a four-step problem solving plan, correlation between examples and exercises, word problems within each lesson, reading and vocabulary support, and student-centered activity labs.

The Glencoe series is a comprehensive program, including all of the major mathematical content strands and mathematical processes. Main Ideas and Key Concepts are highlighted in the Student Edition. The Teacher Edition includes Focus on Mathematical Content for each chapter. This section provides Big Ideas, which explains why the content is important in this chapter and in later studies, and Lesson Summary features, which outline the mathematical ideas of each lesson in the chapter.

The Glencoe series is carefully aligned with national standards. *Pre-Algebra* is also correlated with the NCTM Curriculum Focal Points for Grades 7 and 8. Each curriculum contains Standardized Test Examples and practice problems like those found on standardized tests—within the lessons, the Mid-Chapter Quiz, and the Practice Chapter Test. A separate chapter in the Student Handbook, Preparing for Standardized Tests, provides comprehensive guidance.

The Glencoe curriculum is coherent and coordinated. It is vertically aligned between courses and vertically aligned at the lesson level. The Teacher Edition includes a detailed description of vertical alignment for each chapter. The Focus step in each lesson includes the vertical alignment for that lesson. In the Student Edition, Get Ready for the Chapter provides both a diagnostic quiz and a review of related concepts that were previously studied.

SE - Student Edition **TE - Teacher Edition**

Curriculum Characteristic	Selected Examples			
	Pre-Algebra	Algebra 1	Geometry	Algebra 2
Balance—Understanding and Fluency	SE pp. 418, 427, 429, 455 TE pp. T6–T8	SE pp. 77, 106, 204, 215, 216, 337 TE pp. T6–T8	SE pp. 6, 54, 67, 168, 381, 499 TE pp. T6–T8	SE pp. 78, 194, 247, 260, 388 TE pp. T6–T8
Comprehensive	SE p. 420 TE pp. T12, 416E	SE p. 471 TE pp. T12, 468E	SE p. 497 TE pp. T12, 494E	SE p. 160 TE pp. T12, 160E
Alignment with Standards	SE pp. 138, 174, 809 TE pp. T5, 24A	SE pp. 214, 218, 756 TE p. T5	SE pp. 504, 524, 841 TE p. T5	SE pp. 564, 662, 941 TE p. T5
Coordination and Coherence	SE p. 291 TE pp. 24B, 26	SE p. 185 TE pp. 184B, 196	SE p. 141 TE pp. 140B, 142	SE p. 561 TE pp. 560B, 629

Educational Principles

Professional Development

Research shows that to be effective, teachers need to fully understand the concepts that make up mathematics and underlie mathematical procedures. They need to understand the interconnections among topics (Mewborn, 2003; Swafford, Jones, & Thornton, 1997).

Many teachers lack a conceptual understanding of mathematics, although they have strong procedural knowledge (Mewborn, 2003; Even & Tirosh, 1995).

Teachers need to be actively engaged in learning new ideas and to translate what they learn into classroom practice (Mewborn, 2003; Shifter, 1998).

Teachers must listen to students' reasoning and be able to make sense of students' mathematical thinking (Mewborn, 2003; Thompson & Thompson, 1994, 1996).

Studies show that collegial support for teachers working to improve instructional practices is valuable, but it must be accompanied by a strong focus on mathematical content, students' thinking, and curriculum (Mewborn, 2003).

Teachers need to learn and use three key classroom practices for doing and learning mathematics:

- Supporting discourse;
- Establishing norms, such as what counts as a mathematical explanation; and
- Building relationships, which includes understanding students' identity and culture (Franke et al., 2007).

The Glencoe curriculum provides many opportunities for professional development. Learning opportunities through video, online activities, and on-site instructions are available and fully aligned for *Pre-Algebra* through *Algebra 2*.

This curriculum provides abundant support in clear, easy-to-use Teacher Editions with planning and pacing suggestions, background mathematics information, and vertical alignment charts. Features such as *Teacher Tips, Tips for New Teachers*, and *Focus on Mathematical Content* help teachers understand concepts and student thinking.

What the Research Says explains related research. *Reading and Writing in Mathematics* and *Project CRISS*^{SM} strategies guide teachers in helping students.

Differentiated Instruction

Research demonstrates that participation (what students do and become within the classroom community) and identity (students' perception of themselves and others' perception of them as doers of mathematics) are two critical aspects of classroom practices. (Diversity…, 2007).

Teachers need to know how children think in mathematics in order to make appropriate instructional decisions based on what each child knows and can do (Carey, Fennema, Carpenter, & Franke, 1995).

Studies show that students with learning disabilities benefit from activities-based, hands-on learning, as well as reasoning and problem-solving experiences. Strategies may need to be modified by providing explicit instruction, structuring activities, and stressing vocabulary acquisition (Goldman et al., 1998; Miller & Mercer, 1998).

Effective practices for teaching students with difficulties in mathematics (special education and low-achieving students) include systematic and explicit instruction, student think-alouds, visual and graphic depictions of problems, structured peer-assisted learning in heterogeneous ability groupings, and formative assessment data provided to teachers and to students (Gersten et al., 2006).

Effective strategies for students with learning disabilities include using advance organizers to clarify the lesson's purpose; reviewing major concepts frequently; teaching generalization and application; modeling procedures at a slow pace with extra clues; presenting new skills using concrete materials, then pictures, and finally abstract explanations; providing extra practice in small steps with guidance; giving clear directions before practice; and checking for error patterns (Bishop & Forgasz, 2007).

Research has shown that specific strategies encourage girls in mathematics, including:

- "Teach students that academic abilities are expandable and improvable;
- Provide prescriptive, informational feedback;
- Expose girls to female role models who have succeeded in math and science;
- Create a classroom environment that sparks initial curiosity and fosters long-term interest in math and science; and
- Provide spatial-skills training" (Halpern et al., 2007).

The Glencoe curriculum includes extensive resources to help teachers provide differentiated instruction. Each chapter lists Teacher Resources in a chart with level codes: BL, OL, AL, ELL. Many blackline masters are also available in Spanish. *Real-World Careers* features include women who use mathematics.

Leveled Resources provide worksheets for each lesson to meet individual needs: Lesson Reading Guide, Study Guide and Intervention, Skills Practice, Practice, Word Problem Practice, Enrichment, and Graphing Calculator and Spreadsheet Activities. Each lesson provides teachers with a chart of *Differentiated Homework Options*, for BL, OL, and AL levels. Each chapter (*Pre-Algebra, Algebra 1*, and *Geometry*) includes *Options for Differentiated Instruction*, for ELL, AL, SS (Struggling Students), and SN (Special Needs) and provides teachers with information on the *Quick Review Math Handbook* for intensive intervention. Additional resources are available online and on CD-ROM (*StudentWorks Plus*™.)

Assessment

Research has demonstrated that collecting assessment data from students, both formally and informally, is a key to shaping effective instruction (Brimijoin et al., 2003). This is called *data-driven assessment*.

An extensive research review of 250 journal articles concluded that stronger formative assessments produce significant learning gains (Wilson & Kenny, 2003; Black & Wiliam, 1998a; Black & Wiliam, 1998b).

The use of day-to-day formative assessment is one of the most powerful ways of improving learning in mathematics (Wiliam, 2007). Formative assessment includes determining where learners are in their learning, where they are going, and how they will get there (Wiliam & Thompson, 2007).

Research shows that to effectively use formative assessment, teachers should clarify, share, and understand goals and criteria for success; use classroom discussions, questions, activities, and tasks that elicit evidence of students' learning; provide feedback that tells students how to improve; help students learn to help each other; and help students become responsible for their own learning (Wiliam, 2007).

Effective feedback is focused on the mastery of the task, given soon after the task, and specific and related to students' needs (Marzano et al., 2001; Crooks, 1988).

Frequent, short tests are better than infrequent, long tests. New learning should first be tested within about a week of its first encounter (Black & Wiliam, 1998b).

Performance improves with frequent testing to a certain point, but additional tests have little extra impact (Wilson & Kenny, 2003; Bangert-Drowns, Kulik, & Kulik, 1991).

Throughout the Glencoe series, numerous Diagnostic, Formative, and Summative Assessments are offered. Data-driven decision making provides frequent and meaningful assessment of student progress within the curriculum structure and teacher support materials. Teachers can also create and customize their own assessments using *ExamView Assessment Suite* software.

Initial assessment at the beginning of the year using the *Diagnostic and Placement Tests* booklet assesses students' prior knowledge. *Get Ready for the Chapter* and *Get Ready for the Next Lesson* in the Student Edition checks on students' prior knowledge, while the Teacher Edition provides *Intervention Options* based on the results.

Opportunities for formative assessment are described in the Teacher Edition Assessment Planner for each chapter, listing formative and summative assessment options.

The Student Edition includes *Check Your Progress, Check Your Understanding, Mid-Chapter Quiz*, and *Study Guide and Review* for students to assess their progress. The Teacher Edition for each lesson includes *Step 4 (Assess)* of the Teaching Plan. *Data-Driven Decision Making* charts accompany the *Mid-Chapter Quiz* to help teachers modify instruction based on student performance. Self-check quizzes are available at the course Web site. Students learn to assess their own work in *Find the Error* exercises.

Summative assessments include *Chapter Practice Test* and *Standardized Test Practice*. The Teacher Edition includes *Data-Driven Decision Making* charts for the Chapter Tests. Alternate, leveled chapter tests in six forms are available in the *Chapter Resource Masters* (CRM). Six forms of *Chapter Tests*, four *Quizzes, Vocabulary Test, Extended-Response Test*, and *Standardized Test Practice* are also available.

Technology

Research shows that technology use increases *representation fluency*, the ability to move between representations while carrying the meaning from one representation to another. It does this by providing multiple representations that are linked and interactive (Zbiek et al., 2007).

When technology is used in classrooms, teachers more often tend to take on the role of counselor, providing mathematical assistance when students request it (Zbiek et al., 2007).

Students' behaviors vary when using technological tools. Some students always use them judiciously; some use the tools capriciously and inappropriately. Most students can be educated to work judiciously with these tools (Zbiek et al., 2007; Ball & Stacey, 2005).

Research results on calculator use are generally positive (Apthorp et al., 2001). Research on graphing calculator use also shows positive effects on graphing ability, conceptual understanding of graphs, and relating graphs to other representations (Grouws & Cebulla, 2000).

Calculator use enhances learning arithmetical concepts and skills, problem solving, and attitudes of students (Hiebert, 2003). Most graphing calculator studies have *not* found negative effects on basic computation skills or factual knowledge (Grouws & Cebulla, 1999).

With calculators, teachers tend to ask more high-level questions; students are more actively involved in asking questions and conjecturing (Grouws & Cebulla, 1999).

Research on computer use indicates that application software, such as spreadsheets and graphing tools, can promote learning (Grouws & Cebulla, 2000). The most effective use of instructional software for mathematics occurs when meaning and understanding are emphasized, mathematical skills are embedded in context, and connections are made between subject areas and real-life experiences (Reyes et al., 1999). Software like *Geometer's Sketchpad*, with its interactive, dynamic representations, can help students draw meaning about a mathematical entity if the students reflect on the connection between their actions and the change in representations (Zbiek et al., 2007).

The Glencoe curriculum offers fully integrated technology resources for teachers, students, and parents online and on CD-ROMs and DVDs, providing support for differentiated instruction, alternate teaching approaches, additional assessment opportunities, and motivating activities.

Students in *Pre-Algebra* are introduced to graphing calculators at the beginning of the year. The curriculum includes several *Graphing Calculator Labs* and *Spreadsheet Labs*.

Algebra 1 contains 14 *Graphing Calculator Labs*, ranging from linear functions to exponential functions to scatter plots and a *Spreadsheet Lab* on weighted averages.

In *Geometry*, *Geometry Software Labs* use *Geometer's Sketchpad* software. *Graphing Calculator Labs* explore a variety of concepts and *Spreadsheet Labs* explore volume and angles in polygons.

In *Pre-Algebra*, *Algebra 1*, and *Geometry*, *What's Math Got to Do With It? Real Life Math Videos* on DVD show students how math is used in everyday situations.

Algebra 2 contains 17 *Graphing Calculator Labs*, including function graphing, limits, solving equations, matrices, and modeling data. It offers three *Spreadsheet Labs*, including amortizing loans.

Throughout the series, *Concepts in Motion*, referenced in the Teacher Edition, provide online illustrations of key concepts. *MindJogger Plus* on DVD reviews concepts in a game-show format with student teams. *Internet Resources* for each chapter are listed in a table in the Teacher Edition.

SE - Student Edition **TE - Teacher Edition**

Educational Principles	Selected Examples			
	Pre-Algebra	Algebra 1	Geometry	Algebra 2
Professional Development	TE pp. T10–T23, 24A–24H, 31, 45	TE pp. T10–T23, 4A–4H, 22, 38	TE pp. T10–T23, T26, 4A–4H, 61, 98	TE pp. T10–T21, T24, 4A–4F, 40, 43
Differentiated Instruction	TE pp. T20–T21, 24C, 24H, 26, 28, 605	TE pp. T20–T21, 4C, 4H, 6, 18, 201	TE pp. T20–T21, 4C, 4H, 17, 20–21, 338	TE pp. T18–T19, 4C, 8, 19, 329
Assessment	TE pp. T14–16, 123, 146, 169–175	TE pp. T14–16, 141, 164, 177–183	TE pp. T14–17, 141, 164, 191–197	TE pp. T13–15, T25, 115, 137, 153–159
Technology	SE pp. 10–11, 337, 390 TE pp. T8, T11, T17, 226C–D	SE pp. 203, 252 TE pp. T8, T11, T17, 226C–D	SE pp. 155, 324 TE pp. T8, T11, T16, 378C–D	SE pp. 168, 293 TE pp. T8, T11, 496C–D

Instructional Strategies

Balanced Instruction

Research shows that teachers cannot simply transfer knowledge to students by lecturing. Students have to take an active role in their own learning. To accomplish this, mathematics programs must include ample opportunity to explore, question, discuss, and discover. The learning experience should include a balance of discovery learning and explicit instruction. Students need to learn how to grapple with problems and construct conceptual knowledge (Pressley, Harris, & Marks, 1992; Shulman & Keislar, 1996). Explicit instruction occurs when teachers and textbooks clearly explain problem-solving strategies to students (Duffy, 2002). In geometry instruction, a combination of expository and discovery methods has been found to be effective, especially when manipulatives are used to help students represent and comprehend geometric concepts (Klausmeier, 1992).

Glencoe's curriculum offers a balanced approach of real-world applications, hands-on labs, direct instruction, writing exercises, and practice that enables students to develop both conceptual understanding and procedural knowledge. Lessons begin with real-world problems to solve, and then students use multiple representations to explore new concepts. *Algebra Labs* and *Geometry Labs* provide carefully designed opportunities for discovery. *Graphing Calculator Labs* and *Spreadsheet Labs* use technology to promote discovery of patterns and relationships.

Use of Prior Knowledge

Prior-knowledge strategies help students retrieve information stored in their long-term memories to learn new, related information. These strategies include recalling remembered information, asking questions, elaborating on textbook and teacher information, and referring students to other meaningful information. Asking students to use prior knowledge may remind them of information already in their long-term memory that, for some reason, is not easily remembered (Bransford, 1979; Pressley & McCormick, 1995).

Glencoe's series of courses intertwine concepts and continuously refer to material in previous chapters and in students' personal experiences. *Spiral Review* exercises are included in each lesson. *Get Ready for the Next Lesson* reminds students of prerequisite skills. The Teacher Edition includes a *Vertical Alignment* at the beginning of each lesson, so teachers are aware of previous and future related concepts. *Vocabulary Link* features in the Student Editions connect everyday uses and math uses of words, for example, "distribute" and "distributive property." *Study Tips* in the margin include *Look Back* features that help students review previously learned concepts at the point that they will be used.

Practice

Providing students with practice on important tasks has long been considered a successful strategy to improve understanding and memory. Giving students individual feedback on their practice helps them monitor and improve their mathematical learning. Practicing helps students acquire additional information as they work towards understanding and applying mathematical concepts. Research shows that mastering a skill requires focused practice. During practice, students adapt and shape what they have learned. In doing so, they increase their conceptual understanding (Clement, Lockhead, & Mink, 1979; Davis, 1984; Mathematical Science Education Board, 1990; Romberg & Carpenter, 1986).

Mastering a skill requires considerable focused practice. To reach a level of 80% of mastery usually takes about 24 practice sessions. However, the first four practice sessions lead to an approximately 48% level of proficiency (Marzano et al., 2001).

Check Your Understanding exercises in each lesson ensure that students understand the concepts and skills before they practice independently. Teachers can provide feedback before students work exercises on their own, either in class or as homework. *Differentiated Homework Options* in the Teacher Editions help teachers provide students with appropriate practice. *H.O.T. Problems* give practice in using higher order thinking skills. The *Spiral Review* section in every lesson includes cumulative practice. Each lesson also contains *Standardized Test Practice*. Each chapter provides two pages of *Standardized Test Practice* problems that enable students to practice skills and at the same time to gain experience in taking standardized tests. Additional worksheets (Skills Practice, Practice, Word Problem Practice) and online practice quizzes are also available, as is the *Extra Practice* section in the Student Handbook.

Note Taking

In the process of note-taking, students identify important items from reading and write them in an organized format. As they write and draw notes, students see relationships within the information. Notes should not be verbatim; note-taking is most valuable when students learn to select important points (Marzano et al., 2001; Bretzing & Kulhary, 1979). Students benefit from learning various note-taking formats, since there is no single correct way to take notes (Marzano et al., 2001). Studies show that using notes as a study guide for tests is an effective form of review (Marzano et al., 2001).

When study skills, such as note-taking, are taught within the teaching of content, they promote learner activity and improve metacognition (Hattie et al., 1996; Robinson & Kiewra, 1995).

Each chapter in the Glencoe mathematics series includes instructions for study organizers, called *Foldables*®, created by Dinah Zike. Study organizers are handmade paper booklets. The *Foldable*, designed to fit each chapter's content, guides students in choosing the important concepts and recording them in an organized format. Since students make their own three-dimensional *Foldables* as well as enter the notes, they feel a sense of ownership. *Foldables* also help students review the chapter's concepts.

The *Study Guide and Review* feature at the end of each chapter consists of notes plus worked examples and practice exercises. The *Lesson-by-Lesson Review* presents a clear picture of the important concepts in each lesson and provides students with a model for taking notes. *Project CRISS Study Skills* include many note-taking strategies, such as Venn diagrams. The *Notables*™ *Interactive Study Notebook with Foldables* for each chapter helps students take and organize notes.

Cooperative Learning

Cooperative learning occurs when students work in pairs or groups of three or four to complete tasks. Research shows that cooperative learning provides practice at valuable skills, such as positive interdependence, face-to-face interactions, individual and group accountability, interpersonal skills, and group processing (Johnson & Johnson, 1999). Cooperative learning has a highly positive effect when compared with strategies in which students compete with each other and strategies in which students work on tasks individually (Marzano et al., 2001; Johnson, Maruyama, Johnson, Nelson, & Skon, 1981). A balance of cooperative learning and individual learning is needed, however, because students need time to practice skills independently (Anderson, Keder, & Simon, 1997).

Glencoe's mathematics series provides a mix of individual and cooperative learning opportunities. The Teacher Editions include a detailed section on Cooperative Group

Strategies, helping teachers implement this strategy. A variety of labs—*Algebra Labs, Graphing Calculator Labs, Geometry Labs,* and *Spreadsheet Labs*—located throughout the texts, direct students to work with other students in carefully structured activities. In *Algebra Labs*, students work together in small groups to collect data, analyze data, and make conjectures. In lessons where it is appropriate, *Tips for New Teachers: Working Together* suggest cooperative learning opportunities. *Differentiated Instruction: Interpersonal Learners* describes additional learning activities where students work in pairs.

Identifying Similarities and Differences

Comparison and classification skills are vital in mathematics and science. Students identify similarities and differences to determine how the current problem and previously solved problems are alike and different. Research tells us that comparing and classifying are effective ways to identify similarities and differences (Chen, 1996; English, 1997; Newby et al., 1995). The most effective methods of working with similarities and differences are to have students identify similarities and differences on their own (Chen, 1996; Mason & Sorzio, 1996), use graphic and symbolic representations as well as words (Mason, 1994), and begin with concrete examples and then move towards abstract knowledge (Reeves & Weisberg, 1993).

In Glencoe's mathematics series, students often explain the difference between two related concepts, such as a sequence and a series. They compare and contrast graphs or equations that belong to a family of graphs or equations, respectively. In *H.O.T. Problems, Which One Doesn't Belong?* exercises are classification problems, and some Reasoning questions ask students to compare two items. Students also interpret and create Venn diagrams and other diagrams showing classification, such as various types of polygons.

Reading and Writing in Mathematics

Reading and writing are two of the major activities through which students acquire mathematical understanding. Their importance increases as grade level increases.

Reading mathematics requires unique knowledge and skills, since mathematical writing is dense, terse, and comprised of symbols as well as words. (Barton & Heidema, 2002). Reading helps students understand concepts, work problems, organize ideas, extend their thinking, and view mathematics as a valuable subject (Martinez & Martinez, 2001).

Teachers should model for students how to monitor and reflect on their reading. Modeling by thinking aloud is effective (Barton & Heidema, 2002). Teachers can model reading strategies by reading a problem aloud, paraphrasing it, and talking through how to figure out word meanings by asking if everyone is clear on a word's meaning and by asking questions about the meaning of the problem (Kenney et al., 2005). Reading word problems requires special skills: understanding the problem situation, identifying the main idea, and extracting relevant information (Musthafa, 1996).

Mastering vocabulary is fundamental to mathematical understanding. Students struggle with mathematics vocabulary because of the overlap between mathematical and everyday English usage, the use of mathematical symbols (Fuentes, 1998), and concepts embedded within other concepts (Barton & Heidema, 2002). Direct instruction on vocabulary words increases student achievement (Stahl & Fairbanks, 1986). Graphic organizers, such as concept maps, help students learn vocabulary (Monroe, 1997). Research has shown that repetition and multiple exposures to vocabulary words are effective (National Reading Panel, 2000).

Textbook features—organization, presentation, coherence—have a direct impact on reading comprehension (Barton & Heidema, 2002). Characteristics that assist reading comprehension include a page layout that makes the organization of the content evident, a consistent pattern within each lesson or chapter, and explicit instruction on the text structure (Barton & Heidema, 2002; Dickson, Simmons, & Kameenui, 1995). Teachers should introduce students to the visual cues used in their textbook—colors, fonts, graphic symbols, text boxes—and model how to use features of their textbook (Barton & Heidema, 2002).

Writing in mathematics includes writing study notes, vocabulary definitions, explanations, descriptions, predictions, and justifications, along with attitudes, confusions, and questions. Writing helps students make sense of mathematics (Countryman, 1992). The thinking skills involved in writing are major aspects of doing mathematics. Through writing, students clarify and organize their thinking. Writing reinforces what students have learned, helping to retain it in long-term memory.

Research shows that frequent, regular writing in mathematics class leads to improved quality of writing, improved student attitudes toward mathematics, and teachers' increased insight into their teaching (Miller & England, 1989). Writing is effective in helping learn mathematical concepts. Writing by students gives teachers valuable information on student learning (Powell, 1997). When students write explanations about how to solve numerical problems, they are doing much more than acquiring content and demonstrating mastery of a benchmark. They are also communicating and problem solving (Urquhart & McIver, 2005).

The Student Editions of Glencoe's mathematics series offer a consistent book layout and a clear, consistent lesson structure. The consistent page layout makes effective use of color, graphics, and fonts. Concepts are highlighted in shaded *Key Concept* and *Theorem* boxes, in words, symbols, and models and are summarized in the *Study Guide and Review*, with examples and exercises.

Reading Math margin notes in the Student Edition focus attention on reading mathematical symbols. Full-page *Reading Math* activities help students master mathematical language and symbols. Dinah Zike's *Foldables* in each chapter help students organize their learning through reading and writing.

The *Teacher Handbook* includes *Reading Strategies*, which discusses activating prior knowledge, setting a purpose, developing vocabulary, taking notes, and summarizing.

Writing in Math questions in every lesson require students to use critical thinking skills. *Project CRISS*SM *Study Skill* features in each chapter of the Teacher Edition include writing skills, such as note taking or writing the steps in a mathematical process.

Visual Literacy

Visuals used in conjunction with verbal description increase student understanding and improve memory of relationships and properties. Visuals are often the only way to effectively communicate ideas that explain central concepts needed to understand algebra and geometry. Research shows that students are better able to organize and group concepts when visuals illustrate different and common characteristics (Hegarty, Carpenter, & Just, 1991). Also, the mental images that high-quality visuals encourage are an indispensable tool for recalling information, especially compared to information presented with only text or lower-quality visuals (Willows & Houghton, 1987).

Glencoe's mathematics series includes high-quality charts, tables, graphs, drawings, and photographs throughout the text. The eye-catching photographs that begin each unit illustrate the *Cross-Curricular Project*. The photographs at the beginning of each chapter portray the *Real World Link* that relates to the main concepts in that chapter.

A visual accompanies each lesson's *Get Ready for the Lesson* feature, drawing student attention and helping communicate the concept. Visuals in lab activities help students understand the activities and begin work. Drawings of algebra tiles in *Algebra Labs* help make connections between concrete tiles and abstract symbols. Photographs in *Real-World Link* margin features connect math concepts to real situations. Detailed images of graphing calculator screens enable students to use the tool efficiently so they can focus on the concepts.

SE - Student Edition **TE - Teacher Edition**

Instructional Strategy	Selected Examples			
	Pre-Algebra	Algebra 1	Geometry	Algebra 2
Balanced Instruction	SE pp. 418, 424–425, 476, 483 TE pp. T6–T8	SE pp. 77, 145, 365, 370 TE pp. T6–T8	SE pp. 67, 158, 252, 273, 332 TE pp. T6–T8	SE pp. 78, 172, 270, 547, 580 TE pp. T6–T8
Use of Prior Knowledge	SE pp. 296, 518, 524 TE p. 292	SE pp. 189, 299, 301 TE p. 294	SE pp. 165, 323, 358 TE pp. 318, 336	SE pp. 41, 312, 318, TE pp. 312, 204
Practice	SE pp. 466–468, 508–509 TE p. 466	SE pp. 475–477, 522–523 TE p. 475	SE pp. 501–503, 548–549 TE p. 501	SE pp. 503–506, 558–559 TE p. 504
Note Taking	SE pp. 408–411 TE pp. 24G, 24, 76H, 408	SE pp. 410–414 TE pp. 4G, 4, 410, 526H	SE pp. 424–426 TE pp. 4G, 4, 424, 430H	SE pp. 430–434 TE pp. 4, 430
Cooperative Learning	SE p. 383 TE pp. T18–T19, 386, 398	SE p. 186 TE pp. T18–T19, 27, 59	SE pp. 266–267 TE pp. T18–T19, 266, 387, 399	SE pp. 92–93 TE pp. T18–T19, 92–93, 293, 734
Identifying Similarities and Differences	SE pp. 191, 200, 395, 529	SE pp. 46, 284, 478	SE pp. 65, 240, 353	SE pp. 265, 640

Reading and Writing in Mathematics	SE pp. 298, 301, 302, 306, 348 TE pp. T6–T7, T9, T22–23, xiii, 290H	SE pp. 314, 315, 319, 346 TE pp. T6–T7, T9, T22–23, xiii, 292H	SE pp. 331, 340, 358, 367, 369 TE pp. T6–T7, T9, T22–23, xii, 316H	SE pp. 312, 319, 344, 363, 374 TE pp. T6–T7, T9, T22–23, xiii
Use of Visuals	SE pp. 176–177, 178, 273, 395, 406	SE pp. 138–139, 140, 187–189, 555, 583–584	SE pp. 376–377, 378, 387, 412, 464	SE pp. 232–233, 234, 250, 450, 518–519

Mathematical Concepts and Skills

Algebra

Studies show that students have difficulty dealing with equations used in various ways—as a formula ($A = lw$), an equation to solve, ($4x = 20$), an identity ($\sin x = \cos x \cdot \tan x$), a property ($a + b = b + a$), or a function ($y = 3x$) (Chazan & Yerushalmy, 2003; Usiskin, 1988). Language plays a crucial role in algebra (RAND, 2003; Wheeler, 1996).

Learning rule-based procedures without creating meaning for the rules or learning when to use them leads to choosing inappropriate strategies (Kilpatrick, Swafford, & Findell, 2001). However, symbolic fluency is central to algebraic proficiency. Teachers need to motivate students' interest to encourage persistence in practice (RAND, 2003).

Connecting algebra with arithmetic, geometry, and statistics can help build student understanding (RAND, 2003). Research into the use of concrete models for solving equations has had mixed results. Some studies have found balance scales and cups and counters to aid understanding, while others concluded that they were not helpful (Kieran, 2007).

Word problems are easier for students than solving the corresponding equations without context, contrary to what most teachers believe (Nathan & Koedinger, 2000). In solving word problems, students tend to revert to arithmetic problem-solving methods instead of using algebra (Stacy & MacGregor, 1999; Kieran 2007).

A *function* pairs each x-value with a corresponding y-value; this is called the *correspondence* approach. Alternately, a *covariational perspective* emphasizes that for a linear function the values in the x column increase by constant, while values in the y column increase by a different constant. When students begin using function tables, they typically begin by noticing covariation, rather than ordered pairs (Smith, 2003; Confrey & Smith, 1995). The covariation approach is useful in modeling (Smith, 2003).

Teaching functions should build on students' prior knowledge, starting with a simple familiar context and having students express concepts in their own language; develop conceptual understanding, procedural fluency, and connected knowledge; and help students become self-regulating problem solvers, using multiple strategies and representations (Kalchman & Koedinger, 2005).

Student understanding is increased when functions are defined as relationships between variables, introduced earlier, used to model concrete situations, and expressed in various representational systems (O'Callaghan, 1998). Students who work with multiple representations of functions develop a more comprehensive understanding of them (Leinhardt, Zaslavsky, & Stein, 1990).

Graphing calculators provide multiple representations. The use of graphing technology in algebra for solving problems and interpreting symbolic representations is an effective approach However, the use of technology in algebra does not remove the need for paper-and-pencil techniques (Kieran, 2007). Many students confuse the algebraic and geometric aspects of slope, scale, and angle. Scale changes for linear graphs result in visual changes, while the slope remains constant (Zaslavsky, Sela, & Leron, 2002).

In the Glencoe mathematics series, functions are defined as relations of ordered pairs. Students explore functions as tables of values, mappings, and graphs. Writing and solving equations is emphasized with each lesson offering a balance of both extensive skills practice and *H.O.T. Problems* (Higher Order Thinking).

In *Pre-Algebra*, function notation is emphasized in a *Reading Math* feature. Slope is introduced in a lesson on rate of change, in a hands-on *Algebra Lab*, and in a *Graphing Calculator Lab*. Other *Algebra Labs* use algebra tiles to help students solve equations. Writing and solving equations is emphasized in *Reading Math* features, in lessons, and in word problems throughout the curriculum. Three chapters in Unit 4, *Applying Algebra to Geometry*, reinforce concepts from both topics and help students connect them.

In *Algebra I*, relations and functions are defined in terms of ordered pairs and represented as ordered pairs, tables, graphs, and mappings. *Graphing Calculator Labs* enable students to explore families of functions. Lines of fit connect algebra to statistics; equations of parallel and perpendicular lines tie algebra to geometry. Students graph ordered pairs of data to investigate real-world functions. *Multiple-Step Word Problems* throughout the lesson exercises provide practice in using algebra in realistic situations.

In *Geometry*, connections to algebra occur in many lessons, including the distance formula, coordinate geometry coordinate proofs, vectors, and trigonometry.

In *Algebra 2*, seventeen *Graphing Calculator Labs* explore function graphing, limits, solving equations, matrices, and modeling data. This course includes a wide array of connections to other areas of mathematics: scatter plots, linear programming, matrices, complex numbers, logarithmic functions, modeling motion, exponential functions, conic sections, sequences and series, mathematical induction, normal distribution, and trigonometric functions.

Geometry and Measurement

The van Hiele theory (van Hiele 1986; van Hiele, 1959/1985/2004), which defines sequential levels of reasoning in geometry, is valid and useful in describing students' geometric-concept development (Clements & Batista, 1992; Battista, 2007).

Level 1 Visual – recognize shapes but not think about properties
Level 2 Descriptive/analytic – characterize shapes by their properties
Level 3 Abstract/relational – form definitions; understand and provide arguments
Level 4 Deduction – construct proofs; use definitions, axioms, theorems

Research indicates that the van Hiele levels may not be discrete; students often use more than one level of reasoning (Gutiérrez et al., 1991). The levels may develop simultaneously and at different rates, depending on maturation and instruction (Battista, 2007).

Students already have "fuzzy" concepts for geometric objects. Formal, mathematical definitions may differ from these fuzzy concepts. For example, students may not agree that a square is a rectangle; the fuzzy concept of rectangle does not match the mathematical definition (Battista, 2007). Careful use of geometry vocabulary is important in making clear the distinctions between common usage and mathematical usage (Clements, 2003; Fuys et al., 1988).

Students often confuse drawings with the theoretical geometric objects that the drawings represent (Battista, 2007). For example, they assume that lines that look parallel are parallel or that right triangles must have one horizontal side (Clements & Batista, 1992). Student difficulties in understanding the connection between linear equations and their graphs could be due to lack of understanding of how coordinates are related to length. Students may see coordinates as simply markers, like margin letters on a map (Battista, 2007; Knuth, 2000).

Students who receive training and practice in specific spatial skills (e.g., mental rotation, spatial perspective) improve their performance on mathematics tests (Halpern et al., 2007).

Understanding area and volume measure requires knowing: 1) what area/volume is and how it is conserved, 2) how to measure it by iterating units of area/volume, 3) numerical processes to determine measures for special shapes, and 4) how these processes are expressed in words and symbols (Battista, 2007).

In the Glencoe mathematics series, geometry and measurement concepts and skills are interwoven into each course. In *Pre-Algebra*, three chapters in Unit 4: Applying Algebra to Geometry focus on right triangles, and two- and three-dimensional figures. *Geometry Labs* enable students to work hands-on with geometric figures. *Pre-Algebra* students explore the Pythagorean Theorem and distance formula; they use similar figures in indirect measurement. They work with formulas for area, surface area, and volume; explore similar solids; and draw three-dimensional figures from various perspectives.

In *Algebra 1*, students write monomials as formulas for area and volume of geometric shapes, apply the Pythagorean Theorem and the distance formula, and explore parallel and perpendicular lines in their study of slope.

In *Geometry*, hands-on activities in *Geometry Labs* help students connect concrete and abstract concepts. *Geometry Labs* also provide van Hiele Level 3 activities, such as conjecturing about the angles of triangles. Students use *Geometer's Sketchpad* software in *Geometry Software Labs* for a variety of activities, including making and testing conjectures.

Geometric vocabulary is presented verbally, in symbols, and with figures. *Vocabulary Links* in the margins clarify differences between everyday uses and mathematical uses of words, like *median*. *Reading Math* features in each chapter focus on geometry vocabulary. The symbols and diagram markings used to indicate relationships, such as parallel lines, are clearly stated and emphasized. Definitions, postulates, theorems, and proofs are emphasized and used throughout the course.

Linear equations and slopes of lines are reviewed and extended to equations of perpendicular bisectors and determining that lines are parallel using coordinate geometry.

Right triangle trigonometry provides connections to many real-world problems. Transformations combine coordinate geometry, symmetry, nature, and art. Surface area and volume of solids is explored using various representations: models, nets, isometric views, orthographic drawings, cross sections of solids.

In *Algebra 2*, Chapters 13 and 14 on trigonometric functions and graphs are based on right-triangle geometry. A full chapter on conic sections combines geometry and algebra. Throughout the exercises, geometric problem situations, such as surface area and volume, relate geometry to algebra.

Number and Operations

Number and Operations at the high-school level includes understanding the rational and real number systems and their properties, representing numbers in a variety of ways, understanding how operations relate to each other, computing fluently, and making estimations (NCTM, 2000). It includes the effective organization of this knowledge and the ability to apply it (Donovan & Bransford, 2005).

Research studies show that many students in grades 9 and 10 are unable to classify real numbers as rational or irrational (Fischbein et al., 1995). Other studies demonstrate that understanding fundamental structural properties of a mathematical system improves retention of the system and application to new situations (Fey, 1990).

In *Pre-Algebra*, *Algebra 1*, and *Algebra 2*, students use a Venn diagram to understand the real number system and classify real numbers.

Students use number lines to compare rational numbers and real numbers. Students identify and apply properties of number systems. They learn and apply properties of square roots.

In *Algebra 2*, students extend the number system to include imaginary numbers, the complex number system. The chapter on matrices includes properties of matrix multiplication, identity and inverse matrices, and applications.

Statistics, Data Analysis, and Probability

Studies show that the concept of *mean* develops over many years. The mean should be introduced as a *typical value* or a *fair share*, to build on students' intuitions. Later, the mean should be viewed as a *data reducer*, that is, a summary of data and a *signal amid noise*, or a tool to compare data sets (Shaughnessy, 2007; Watson & Moritz, 2000).

Many students use the mode when the median or mean is more appropriate. Few students understand anything about the median and mean other than how to compute them (Konold & Higgins, 2003; Russell et al., 2002; Cai, 1998).

Research indicates that effective statistics teachers emphasize variability, introduce the comparison of two data sets, build on students' intuitive notions of centers and variation, use proportional reasoning to connect samples and populations, and keep in mind that statistics is different from mathematics; the context of the data is critical (Shaughnessy, 2007).

Concepts of chance and randomness challenge learners of all ages. Students fail to interpret uncertainty in patterns that emerge from repetitions, believe that a person or device (for example, a spinner) can control outcomes, and believe that some sort of order or reason underlies events. Students do not consider sample spaces when determining probabilities, even if they can list the outcomes in the sample space (Jones, Langrall, & Mooney, 2007).

Probability instruction should emphasize building sample spaces (all possible outcomes), connect probability and statistics, introduce probability through data, and use a problem-solving approach (Shaughnessy, 2003). Teachers should take into account student preconceptions and misconceptions, provide instruction in probability reasoning to challenge misconceptions, and use an experimental approach (Jones, Langrall, & Mooney, 2007; Castro, 1998).

The curriculum should include the concepts of chance variation, randomness, independence, calculating probabilities, terminology and language use, critical reasoning and questioning, and interpretation of the context, or the situation (Jones, Langrall, & Mooney, 2007; Gal, 2005).

In *Pre-Algebra*, students compare two data sets using histograms, stem-and-leaf plots, and box-and-whisker plots. Students explore experimental probability using graphing calculators for simulations.

In *Algebra 1*, Chapter 12 offers statistics and probability topics, including sampling, permutations and combinations, probability of compound events, probability distributions, and probability simulations. A *Reading Math* activity focuses on Survey Questions. *Algebra Lab: Simulations* involves students in experimenting with probability situations. Sample space is the focus of Lesson 12-2, *Counting Outcomes*.

In *Geometry*, students relate area to probability and explore circle graphs.

In *Algebra 2*, Chapter 12, *Probability and Statistics*, includes multiplying and adding probabilities, statistical measures, the normal distribution, exponential and binomial distribution, binomial experiments, and sampling and error. Word problems focus on choosing among median, mean, and mode to justify a claim. *Algebra Labs* on Simulations and on testing hypotheses provide explorations. A *Reading Math* activity on permutations and combinations is included.

Reasoning

Research tells us that students can show reasoning ability when: 1) they have sufficient knowledge, 2) the task is understandable and motivating, and 3) the context is familiar and comfortable (Kilpatrick et al., 2001; Alexander et al., 1997). Students engage in reasoning, explanations, and justification when they are in a classroom environment that supports these activities. Creating this classroom atmosphere requires explicit effort by the teacher (Yackel & Hanna, 2003; Lampert, 1990).

Explaining and justifying ideas not only helps students clarify and improve reasoning skills, it also increases their conceptual understanding (Kilpatrick et al., 2001; Marzano et al., 2001; Maher & Martino, 1996).

Mathematical language is the foundation of mathematical reasoning (Ball & Bass, 2003). Teachers should model the process of reasoning—building on others' ideas, using the common knowledge base, and using mathematical language carefully (Ball & Bass, 2003).

Students need to learn what constitutes justification in mathematics (Yackel & Hanna, 2003). They need to explain their hypotheses and conclusions, preferably in writing, to deepen their understanding (Marzano et al., 2001).

Students need to justify procedures repeatedly over time, relating new concepts and procedures to those they already understand (Kilpatrick et al., 2001). They develop their concepts of justification and proof over many years, beginning with "making sense." Mathematical reasoning is a process rather than a skill (Yackel & Hanna, 2003; Maher & Martino, 1996).

Deductive reasoning in geometry occurs at Level 3 of the van Hiele model when students can logically connect properties of concepts. The truth of a statement is connected to the accepted truth of other statements (Clements, 2003; Senk 1989). Students should first learn to create informal proofs, chains of informal correct conclusions starting from valid premises (Yackel & Hanna, 2003; Blum & Kirsch, 1991). Formal, mathematical proofs differ in several ways from reasoning in everyday life, which makes it difficult to learn to create proofs (Yackel & Hanna, 2003; Fishbein 1999).

Interactive geometry software can help motivate and enable students to investigate, generalize, and conjecture. It may in some cases blur the distinction between empirical evidence and proof (Yackel & Hanna, 2003; Dreyfus & Hadas, 1996; Chazan, 1993).

Throughout Glencoe's mathematic series, reasoning problems in *H.O.T. Problems* (Higher Order Thinking) occur in each chapter. These include compare and contrast exercises and tell whether a statement is always, sometimes, or never true. *Writing in Math* exercises in *H.O.T. Problems* include many opportunities for students to explain their reasoning.

In *Pre-Algebra* and *Algebra 1*, students explore deductive reasoning, counterexamples, conjectures, and inductive reasoning. In *Algebra 1*, a lesson on *Logical Reasoning and Counterexamples* is included in Chapter 1. Several *Reading Math* activities involve reasoning and proof. *Graphing Calculator Labs* involve predicting the appearance of graphs of equations and describing how changes in equations affect graphs.

In *Geometry*, Chapter 2, *Reasoning and Proof*, provides the introduction to methods of proof used throughout the curriculum. Each of these proof topics is studied extensively: inductive reasoning and conjecture; conditional statements; deductive reasoning; postulates; paragraph proofs; algebraic proof; geometry proof; two-column, formal proofs; and coordinate proofs. Reasoning problems in *H.O.T. Problems* occur in each chapter. Some *Reading Math* activities relate to reasoning and proof. *Writing In Math* questions include many opportunities to explain, for example, how to balance a paper triangle on a pencil point. *Graphing Calculator Labs* involve making and testing conjectures.

In *Algebra 2*, proofs are frequently included in the lessons, as appropriate, especially in the trigonometry chapters. Many Reasoning exercises in *H.O.T. Problems* ask students to create proofs related to the lesson content. Additional *Reasoning* exercises and *Writing in Math* exercises focus on reasoning.

SE - Student Edition **TE - Teacher Edition**

Mathematical Concepts and Skills	Selected Examples			
	Pre-Algebra	Algebra 1	Geometry	Algebra 2
Algebra	SE pp. 134, 152–153, 359, 365–371, 383, 390	SE pp. 59, 125, 143–162, 210, 227	SE pp. 21, 251, 459	SE pp. 346, 567–570
Geometry and Measurement	SE pp. 458, 485, 499, 531, 582	SE pp. 187–189, 236–237, 362, 549–560	SE pp. 49, 58–59, 67, 142, 149–150, 156–187, 209, 358, 464–469, 496–532, 680–764	SE pp. 24–25, 367, 560–611, 756–815, 822–836
Number and Operations	SE pp. 78–110, 209, 464–474, 491	SE pp. 38, 46–52, 60, 528–540	SE p. 111	SE pp. 11–17, 34, 43, 169–193, 261
Statistics, Data Analysis, and Probability	SE pp. 273–280, 624–695	SE pp. 642–643, 649–654, 678	SE pp. 556–557, 665–670	SE pp. 684–744
Reasoning	SE pp. 207, 238, 249, 394	SE pp. 38, 39–44, 89, 218, 453, 478–479	SE pp. 36, 78–131, 190, 277, 295	SE pp. 26, 395, 405, 525

References

Alexander, P.A., White, C.S., & Daugherty, M. (1997). Analogical reasoning and early mathematics learning. In L.D.English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 117–147). Mahwah, NJ: Erlbaum.

Anderson, J.R., Reder, L.M. & Simon, H.A. (1997). *Applications and misapplications of cognitive psychology to mathematics education.* Unpublished manuscript. Pittsburgh, PA: Carnegie Mellon University.

Apthorp, H.S., Bodrova, E., Dean, C.B., & Florian, J. (2001) *Noteworthy perspectives: Teaching to the core—Reading, writing, and mathematics.* New York: McREL.

Ball, D.L. & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W.G. Martin, & D. Schifter (Eds.) *A research companion to principles and standards for school mathematics,* Reston, VA., pp27–44.

Bangert-Drowns, R.L, Kulik, J.A., & Kulik, C-L.C. (1991). Effects of frequent classroom testing. *Journal of Educational Research,* 85, 89–99.

Barton, M.L. & Heidema, C. (2002). *Teaching reading in mathematics: A supplement to Teaching reading in the content areas: If not me, then who? 2nd edition.* Alexandria, VA: Association for Supervision and Curriculum Development.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F.K. Lester (Ed.). *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics*. Charlotte, NC: Information Age Pub.

Bishop, A.J. & Forgasz, H.J. (2007). Issues in access and equity in mathematics education. In F.K. Lester, (Ed.) *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics*. Charlotte, NC: Information Age Pub.

Black, P. & Wiliam, D. (1998a). Assessment and classroom learning. *Assessment in Education*, 5 (1): 7-74

Black, P. & Wiliam, D. (1998b). Inside the black box: Raising standards through classroom assessment. *Phi Delta Kappan*, 80 (2): 139–148.

Blum, W. & Kirsch, A. (1991). Preformal proving: Examples and reflections. *Educational Studies in Mathematics*, 22, 183–203.

Bransford, J.D. (1979). *Human cognition: Learning, understanding, and remembering.* Belmont, CA: Wadsworth Publishing.

Bretzing, B.H. & Kulhary, R.W. (1979, April). Notetaking and depth of processing. *Contemporary Educational Psychology*, 4(2), 123–153.

Brimijoin, K., Marquisee, E., & Tomlinson, C. (2003). Using data to differentiate instruction. *Educational Leadership*. 60:5, 70–72.

Cai, J. (1998). Exploring students' conceptual understanding of the averaging algorithm. *School Science and Mathematics*. 98, 93–98

Carey, D.A., Fennema, E., Carpenter, T.P., & Franke, M.L. (1995). Equity and mathematics education. In W.G. Secada, E. Fennema, & L.B. Adajian (Eds.), New directions for equity in mathematics education. New York, NY: Cambridge University Press.

Castro, C.S. (1998). Teaching probability for conceptual change. *Educational Studies in Mathematics, 35*, 233-254.

Chazan, D. & Yerushalmy, M. (2003). On appreciating the cognitive complexity of school algebra: Research on algebra learning and directions of curricular change. In Kilpatrick, J, Martin, W.G., and Schifter, D., (Eds.) *A research companion to principles and standards for school mathematics*. Reston, VA: NCTM.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24, 359–387.

Chen, Z. (1996). Children's analogical problem solving: The effects of superficial, structural, and procedural similarities. *Journal of Experimental Child Psychology*, 62(30), 410–431.

Clement, J., Lockhead, J., & Mink G. (1979). Translation difficulties in learning mathematics. *American Mathematical Monthly*, 88, 3–7.

Clements, D.H. (2007). Curriculum research: Toward a framework for research-based curricula. *Journal for Research in Mathematics Education*, *38* (1), 35-70.

Clements, D.H. (2003). Teaching and learning geometry. In J. Kilpatrick, W.G. Martin, and D. Schifter (Eds.) *A research companion to principles and standards for school mathematics*, Reston, VA: NCTM.

Clements, D.H. & Battista, M.T. (1992). Geometry and spatial reasoning. In D.A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning.* New York: Macmillan.

Confrey, J. & Smith, E. (1995) Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, 26, 66–86.

Countryman, J. (1992). *Writing to learn mathematics: Strategies that work*. Portsmouth, NH: Heinemann.

Crooks, T.J. (1988). The impact of classroom evaluation practices on students. *Review of Educational Research,* 58(4), 438–481.

Davis, R.B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* Norwood, NJ: Ablex.

Dickson, S.V., Simmons, D.C., & Kameenui, E.J. (1995). *Text organization and its relation to reading comprehension: A synthesis of the research.* (Technical Report No. 17). Eugene, OR: National Center to Improve the Tools of Education, University of Oregon.

Diversity in Mathematics Education Center for Learning and Teaching. (2007). Culture, race, power, and mathematics education. In F.K. Lester, (Ed.) *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics.* Charlotte, NC: Information Age Pub.

Donovan, S. & Bransford, J. (2005). See National Research Council (U.S.)

Duffy, G. (2002). In the case for direct explanation of strategies. In C.C. Block & M. Pressley (eds.) *Comprehension instruction: Research-based best practices.* New York: Guilford Press, 28–41.

English, L.D. (1997). Children's reasoning in classifying and solving computational word problems. In L.D. English (ed.), *Mathematical reasoning: Analogies, metaphors, and images.* Mahway, NJ: Lawrence Erlbaum.

Even, R. & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject matter. *Educational Studies in Mathematics,* 29, 1–20.

Fey, J. (1990). Quantity. In Steen, L.A., *On the shoulders of giants: New approaches in numeracy.* Washington, D.C.: National Academies Press. pp 61–94.

Fischbein, E., Jehiam, R., & Cohen, D. (1995). The concept of irrational numbers in high-school students and prospective teachers. *Educational Studies in Mathematics*, 29, 1.

Fishbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics*, 38(1-3), 11–50.

Franke, M. J., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practices in mathematics. In F.K. Lester, (Ed.) *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics*. Charlotte, NC: Information Age Pub.

Fuentes, P. (1998). Reading comprehension in mathematics. *The Clearing House*, 72(2), 81–88.

Fuys, D., Geddes, D., & Tischler, R. (1988). The Van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education*, Monograph 3.

Gal, I. (2005). Towards "probability literacy" for all citizens: Building blocks and instructional dilemmas. In G.A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning.* New York: Springer.

Gersten, R., Chard, D., Jayanthi, M., & Baker, S. (2006). Experimental and quasi-experimental research on instructional approaches for teaching mathematics to students with learning disabilities: A research synthesis. Center on Instruction: Technical Report 2002-2. Signal Hill, CA: RG Research Group.

Goldman, S.R., Hasselbring, T.S., & The Cognition and Technology Group at Vanderbilt. (1998). Achieving meaningful mathematics literacy for students with learning disabilities. In D.P. Rivera (Ed.), *Mathematics education for students with learning disabilities: Theory to practice.* Austin, TX: Pro-Ed.

Greeno, J.G. (2003). Situative research relevant to standards for school mathematics. In Kilpatrick, J, Martin, W.G., & Schifter, D., (Eds.) *A research companion to principles and standards for school mathematics*. Reston, VA: NCTM.

Grouws, D.A. & Cebulla, K.J. (2000). Improving student achievement in mathematics, Parts 1 and 2, ERIC Digest, SE 064 317, EDO-SE-00-09 and SE 064 318, EDO-SE-00-10. Dec 2000, rev. 2002.

Gutiérrez, A., Jaime, A., & Fortuny, J.M. (1991). An alternative paradigm to evaluate the acquisition of the van Hiele levels. *Journal for Research in Mathematics Education,* 22(3), 237–251.

Halpern, D., Aronson, J., Reimer, N., Simpkins, S., Star, J., & Wentzel, K. (2007). *Encouraging girls in math and science* (NCER 2007-2003). Washington, DC: National Center for Education Research, Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ncer.ed.gov.

Hattie, J., Briggs, J., & Purdie, N. (1996). Effects of learning skills interventions on student learning: A meta-analysis. *Review of Educational Research,* 66(2), 99–136.

Hegarty, M., Carpenter, P.A., & Just, M.A. (1991). Diagrams in the comprehension of scientific texts. In R. Barr, M.L. Kamil, P.B. Rosenthal, & P.D. Pearson (eds.), *Handbook of reading research,* vol. 2. New York: Longman.

Heibert, J. (1999). Relationships between research and the NCTM standards. *Journal for Research in Mathematics Education*, 30, 4.

Johnson, D.W. & Johnson, R.T. (1999). *Learning together and alone: Cooperative, competitive and individualistic learning.* Boston: Allyn & Bacon.

Johnson, D.W., Maruyama, G., Johnson, R.T., Nelson, D., & Skon, L. (1981). Effects of cooperative, competitive, and individualistic goal structures on achievement: A meta-analysis. *Psychological Bulletin*, 89(1), 47–62.

Jones, G.A., Langrall, C.W., & Mooney, E.S. (2007). Research in probability: Responding to classroom realities. In F.K. Lester (Ed.). *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics*. Charlotte, NC: Information Age Pub.

Kalchman, M. & Koedinger, K.R. (2005). Teaching and learning functions. In National Research Council (U.S.), Donovan, S., & Bransford, J. (Eds.) *How students learn: Mathematics in the classroom*. Washington, D.C.: National Academies Press.

Kenney, J.M., Hancewicz, E., Heuer, L., Metsisto, D., & Tuttle, C.L. (2005) *Literacy strategies for improving mathematics instruction.* Alexandria, VA: Association for Supervision and Curriculum Development.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F.K. Lester (Ed.). *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics*. Charlotte, NC: Information Age Pub.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Klausmeier, H.J. (1992). Concept learning and concept teaching. *Educational Psychologist*, 27, 267–286.

Knapp, M.S., Adelman, N.E., Marder, C., McCollum, H., Needels, M.C., Padilla, C., Shields, P.M., Turnbull, B.J., & Zucker, A.A. (1995). *Teaching for meaning in high-poverty schools*. New York: Teachers College Press.

Knuth, E.J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education*, 31(4), 500–508.

Konold, C. & Higgins, T.L. (2003) Reasoning about data. In Kilpatrick, J., Martin, W.G., & Schifter, D., (Eds.) *A research companion to principles and standards for school mathematics*. Reston, VA: NCTM.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal,* 27, 29–63.

Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, 60, 1–64.

Maher, C.A. & Martino, A.M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education*, 27, 194–214.

Marzano, R.J., Pickering, D.J., & Pollock, J.E. (2001). *Classroom instruction that works*. Alexandria, VA: Association for Supervision and Curriculum Development.

Martinez, J.G.R. & Martinez, N.C. (2001). *Reading and writing to learn mathematics: A guide and a resource book*. Boston, MA: Allyn & Bacon.

Mason, L. (1994). Cognitive and metacognitive aspects in conceptual change by analogy. *Instructional Science,* 22(3), 157–187.

Mason, L. & Sorzio, P. (1996). Analogical reasoning in restructuring scientific knowledge. *European Journal of Psychology of Education,* 11(1), 3–23.

Mathematical Science Education Board. (1990) *Reshaping school mathematics.* Washington, DC: National Academy Press.

Mewborn, D.S. (2003). Teaching teachers' knowledge and their professional development. In Kilpatrick, J., Martin, W.G., & Schifter, D., (Eds.) *A research companion to principles and standards for school mathematics*, Reston, VA: NCTM.

Miller, L.D. & England, D.A. (1989). Writing to learn algebra. *School Science and Mathematics,* 89(4), 299–312.

Miller, S.P. & Mercer, C.D. (1998). Educational aspects of mathematical disabilities. In D.P. Rivera (Ed.), *Mathematics education for students with learning disabilities: Theory to practice*. Austin, TX: Pro-Ed.

Monroe, E.E. (1997). *Using graphic organizers to teach vocabulary: How does available research inform mathematics instruction?* (ERIC Document Reproduction Service No. ED 414 256)

Musthafa, B. (1996). *Learning from texts and reading instruction.* ERIC Document Reproduction Service No. ED 395 268.

Nathan, M.J. & Koedinger, K.R. (2000). Teachers' and researchers' beliefs of students' algebraic reasoning. *Journal for Research in Mathematics Education,* 31, 168–190.

National Reading Panel. (2000). *Teaching children to read: An evidence-based assessment of the scientific research literature on reading and its implications for reading instruction: Reports of the subgroups* (NIH Publication No. 00-4754). Washington, DC: National Institute of Child Health and Human Development, U.S. Government Printing Office. http://www.nichd.nih.gov/publications/nrp/smallbook.cfm

National Research Council (U.S.), Donovan, S. & Bransford, J. (Eds.) (2005). *How students learn: Mathematics in the classroom*. Washington, D.C.: National Academies Press.

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Newby, T.J., Ertmer, P.A., & Stepich, D.A. (1995). Instructional analogies and the learning of concepts. *Educational Technology Research and Development*, 43(1), 5–18.

O'Callaghan, B.R. (1998). Computer-intensive algebra and students' conceptual knowledge of functions. *Journal for Research in Mathematics Education*, 29, 21–40.

Powell, A. (1997). Capturing, examining, and responding to mathematical thinking through writing. *Clearing House*, 71(1), 21–25

Pressley, M., Harris, K.R., & Marks, M.B. (1992). But, good strategy instructors are constructivists! *Educational Psychology Review*, 4, 3–31.

Pressley, M. & McCormick, C. (1995). Cognition, teaching, and assessment. New York, NY: HarperCollins College Publishers.

RAND Mathematics Study Panel (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.

Reeves, L.M. & Weisberg, R.W. (1993). On the concrete nature of human thinking: Content and context in analogical transfer. *Educational Psychology*, 13, 245–258.

Romberg, T.A. & Carpenter, T.P. (1986). Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In M.C. Wittrock (ed.), *Handbook of research on teaching*, 3rd Ed. New York: Macmillan.

Russell, S.J., Schifter, D., & Bastable, V. (2002). *Developing mathematical ideas: Working with data*. Parsippany, NJ: Dale Seymour Publications.

Senk, S.L. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education*, 20, 309–321.

Shaughnessy, J.M. (2003). Research on students' understanding of probability. In Kilpatrick, J., Martin, W.G., and Schifter, D., (Eds.) *A research companion to principles and standards for school mathematics*, Reston, VA: NCTM.

Shaughnessy, J.M. (2007). Research on statistical learning and reasoning. In F.K. Lester (Ed.). *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics*. Charlotte, NC: Information Age Pub.

Shifter, D. (1998). Learning mathematics for teaching: From a teachers' seminar to the classroom. *Journal of Mathematics Teacher Education*. 1, 55–87.

Shulman, L.S. & Keislar, E.R. (eds.) (1996). *Learning by discovery: A critical appraisal*. Chicago, IL: Rand McNally & Company.

Smith, E. (2003). Stasis and change: Integrating patterns, functions, and algebra throughout the K–12 curriculum. In Kilpatrick, J., Martin, W.G., and Schifter, D., (Eds.) *A research companion to principles and standards for school mathematics*. Reston, VA: NCTM.

Sowell, E.J. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education*, 20, 498–505.

Stacy, K. & MacGregor, M. (1999). Learning the algebraic method of solving problems. *Journal of Mathematical Behavior* 18, 149–167.

Stahl, S.A. & Fairbanks, M.M. (1986). The effects of vocabulary instruction: A model-based meta-analysis. *Review of Education Research*, 56(1), 72–110.

Stein, M.K., Remillard, J., & Smith, M.S. (2007). How curriculum influences student learning. In F.K. Lester, (Ed.) *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics*. Charlotte, NC: Information Age Pub.

Swafford, J.O., Jones, G.A., & Thornton, C.A. (1997). Increased knowledge in geometry and instructional practice. *Journal for Research in Mathematics Education*, 28, 467–483.

Thompson, A.G. & Thompson, P.W. (1996). Talking about rates conceptually, Part 2: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, 27, 2–24.

Thompson, A.G. & Thompson, P.W. (1994). Talking about rates conceptually, Part 1: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, 25, 279–303.

Tomlinson, C.A. (2001). *How to differentiate instruction in mixed-ability classrooms*. 2nd Ed. Alexandria, VA: Association for Supervision and Curriculum Development.

Urquhart, V. & McIver, M. (2005). *Teaching writing in the content areas*. Alexandria, VA: Association for Supervision and Curriculum Development.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A.F. Coxford & A.P. Schulte (Eds.) *The ideas of algebra*. (1988 Yearbook). Reston, VA: National Council of Teachers of Mathematics.

Van Hiele, P.M. (1986). *Structure and insight: A theory of mathematics education*. Orlando, FL: Academic Press.

Van Hiele, P.M. (1959/1985/2004).The child's thought and geometry. In D. Fuys, D. Geddes, & R. Tischler. (Eds.) *English Translation of Selected Writings of Dina van Hiele-Geldof and Pierre M. van Hiele*. ERIC/SMEAC. In Carpenter, T. P., Dossey, J. A., & Koehler, J. L. (Eds.) (2004). *Classics in mathematics education research*. Reston, VA: National Council of Teachers of Mathematics.

Wheeler, D. (1996). Rough or smooth? The transition from arithmetic to algebra in problem solving. In N. Bednarz & C. Kieran (Eds), *Approaches to algebra: Perspectives for research and teaching*. Dordrecht, Netherlands: Kluwer Academic Publishers.

Wiliam, D. (2007). Keeping learning on track. In F.K. Lester, (Ed.) *Second handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics*. Charlotte, NC: Information Age Pub.

Wiliam, D. & Thompson, M. (2007). Integrating assessment with instruction: What will it take to make it work? In C.A. Dwyer (Ed.). *The future of assessment: Shaping teaching and learning*. Mahwah, NJ: Lawrence Erlbaum Associates.

Willows, D.M. & Houghton, H.A. (1987). *The psychology of illustrations: Basic research*, vol. 1. New York: Springer-Verglag.

Wilson, L.D. & Kenny, P.A. (2003). Classroom and large-scale assessment. In J. Kilpatrick, W.G. Martin, & D. Schifter, (Eds.) *A research companion to principles and standards for school mathematics*. Reston, VA: NCTM.

Yackel, E. & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W.G. Martin, & D. Schifter (Eds.) *A research companion to principles and standards for school mathematics*. Reston, VA: NCTM.

Zaslavsky O., Sela H., & Leron U. (2002). Being sloppy about slope: The effect of changing the scale. *Educational Studies in Mathematics*, 49(1), 119–140.